A semi-Markov model with geometric renewal processes

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Summary

1. Introduction

- Semi-Markov processes are an important tool in modeling real systems.
- Our starting point is the article of Pérez-Ocón and Torres-Castro[1].
- Repairable system with internal and external failures and possible performance decrease after maintenance.
- New calculation method for reliability measures calculation.

[1] Pérez-Ocón and Torres-Castro (2002)

2. Model assumptions

- One component system with two states (up and down)
- Down state refers to maintenance duration period
- Two failure types: internal and external.
- All failures occur independently of each other.
- Internal failure occurs by aging
- External failure arrivals : Homogeneous Poisson Process
- Repairable failure with probability p
- Instantaneous perfect maintenance after non repairable failures or after N repairs of repairable failures.

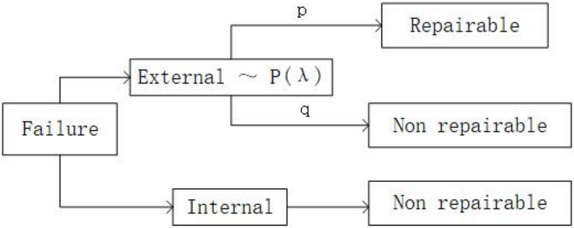


Figure: System Failures

- F the CDF of a new system with finite mean,
- The system deteriorates after each repair
- U_i: lifetime of the system after the i-th repairs with F_i as CDF:
$$F_i(x) = P(U_i \leqslant x) = F(a^i x), x \geqslant 0$$
where a is the operational factor .
- G the CDF of first repair duration with finite mean
- D_i: repair duration after its i-th failures with G_i as CDF:
$$G_i(x) = P(D_i \leqslant x) = G(b^i x), x \geqslant 0$$
where b the repair factor.
- U_i and D_i are independent
- $(U_i)_{i\in\mathbb{N}}$ and $(D_i)_{i\in\mathbb{N}}$ are independent but not identically distributed sequences.

- Up states $U = \{0', 1', 2', ..., N'\}$ where the state 0' is perfect.
- Down states $D = \{0'', 1'', 2'', ..., N-1''\}$,
- One considers a process with state space $E = \{0', 0'', 1', 1'', ..., N-1', N-1'', N'\}$,

Transition of the system is presented in next figure.

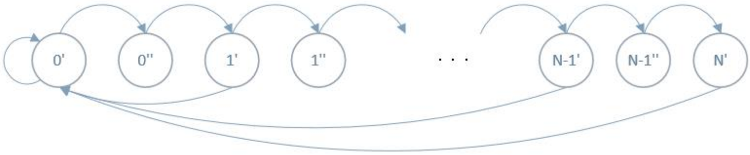


Figure: System Transition

3. Semi-Markov Process

Semi-Markov Process (SMP)

- Semi-Markov Process $(Z_t)_{t\geqslant 0}$, with finite state space E,
- Embaedded Markov Renewable Process (MRP) $(J_n, S_n)_{n\in\mathbb{N}}$,
- $(J_n)_{n\in\mathbb{N}}$ is the Embedded Markov Chain (EMC)
- $S_0 = 0 < S_1 < S_2 < \cdots < S_n < \cdots$ the jump times that $(Z_t)_{t\geqslant 0}$ changes the states:

$$Z_t = J_n, \quad S_n \leqslant t < S_{n+1}$$

- Semi-Markov kernel

$$Q_{ij}(x) = \mathbb{P}(J_{n+1} = j, S_{n+1} - S_n \leqslant x | J_n = i)$$

for $i, j \in E, x \geqslant 0$.

- Standard SMP with $Q_{ii}(x) \equiv 0$.

Semi-Markov Kernel

In standard form, we have the Semi-Markov Kernel is $Q(x), x \geqslant 0$

$$\left(\begin{array}{cccc|cccc}
0 & 0 & \cdots & 0 & \frac{Q_{0'0''}}{1-P_{0'0'}} & 0 & \cdots & 0 \\
Q_{1'0'} & 0 & \cdots & 0 & 0 & Q_{1'1''} & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
Q_{N-1'0'} & 0 & \cdots & 0 & 0 & 0 & \cdots & Q_{N-1'N-1''} \\
Q_{N'0'} & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\
\hline
0 & Q_{0''1'} & \cdots & 0 & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & Q_{N-1''N'} & 0 & 0 & \cdots & 0
\end{array}\right)$$

$$= \left(\begin{array}{c|c} Q_0 & Q_{01} \\ \hline Q_{10} & Q_1 \end{array}\right)$$

Where

Q_0 includes transitions from up states U to up states U;

Q_1 includes transitions from down states D to down states D;

Q_{01} includes transitions from up states U to down states D;

Q_{10} includes transitions from down states D to up states U.

Semi-Markov Kernel

- From i' to state i'': the system fails externally and repaired.

$$Q_{i'i''}(x) = \int_0^x \lambda e^{-\lambda t} p(1 - F(a^i t))dt, \quad i \in \{0, \cdots, N-1\}$$

- Return to initial perfect state: the system has to be replaced by a new one, which means it fails with an internal failure or an non-repairable external failure.

$$Q_{i'0'}(x) = \int_0^x \lambda e^{-\lambda t}(1-p)(1 - F(a^i t))dt + \int_0^x e^{-\lambda t} dF(a^i t), \quad i \in \{0, \cdots, N-1\}$$

- To state $i+1'$, a repairman repairs the system.

$$Q_{i''i+1'}(x) = G(b^i x), \quad i \in \{0, \cdots, N-1\}$$

- After N repairs, we replace the system directly.

$$Q_{N'0'}(x) = \int_0^x \lambda e^{-\lambda t}(1 - F(a^i t))dt + \int_0^x e^{-\lambda t} dF(a^i t)$$

The EMC $(J_n)_{n\in\mathbb{N}}$ gives the successive visited states by the SMP(Z_t) after n^{th} jump. The transition matrix $\mathbf{P} = (P_{ij}; i, j \in E)$ is $P_{ij} = \lim_{x\to\infty} Q_{ij}(x) = Q_{ij}(\infty)$.

$$\mathbf{P} = \left(\begin{array}{ccccc|ccccc}
0 & 0 & 0 & \cdots & 0 & 1 & 0 & 0 & \cdots & 0 \\
P_{1'0'} & 0 & 0 & \cdots & 0 & 0 & P_{1'1''} & 0 & \cdots & 0 \\
P_{2'0'} & 0 & 0 & \cdots & 0 & 0 & 0 & P_{2'2''} & \cdots & 0 \\
\vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\
P_{N-1'0'} & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & P_{N-1'N-1''} \\
1 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 \\
\hline
0 & 1 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 \\
0 & 0 & 1 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & 1 & 0 & 0 & \cdots & 0 & 0
\end{array}\right)$$

$$= \left(\begin{array}{c|c} P_0 & P_{01} \\ \hline P_{10} & P_1 \end{array}\right)$$

The stationary law is $\rho = (\rho_{0'}, \cdots, \rho_{N'}, \rho_{0''}, \cdots, \rho_{N-1''}) = (\rho', \rho'')$

4. Reliability

Initial law of SMP

$$\boldsymbol{\alpha} = (\alpha_{0'}, \cdots, \alpha_{N'}, \alpha_{0''}, \cdots, \alpha_{N-1''}) = (\boldsymbol{\alpha_0}, \boldsymbol{\alpha_1})$$

The distribution function of sojourn time in state i: $H_i(t)$

$$\sum_{j \in E} Q_{ij}(t) = 1 - \overline{H}_i(t)$$

$$\overline{H}(t) = \begin{pmatrix} \overline{H}_{0'}(t) \\ \vdots \\ \overline{H}_{N'}(t) \\ 0 \\ \vdots \\ 0 \end{pmatrix} = \begin{pmatrix} 1 - \frac{Q_{0'0''}(t)}{1-P_{0'0'}} \\ 1 - Q_{0'1'}(t) - Q_{1'1''}(t) \\ \vdots \\ 1 - Q_{N'0'}(t) \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

The mean sojourn time in each state is

$$\boldsymbol{m} = \left(\frac{\boldsymbol{m_0}}{\boldsymbol{m_1}} \right) = \begin{pmatrix} m_{0'} \\ \vdots \\ m_{N'} \\ \hline m_{0''} \\ \vdots \\ m_{N-1''} \end{pmatrix}$$

For Up state, we have

$$\boldsymbol{m_0} = \begin{pmatrix} \int_0^\infty \overline{H_{0'}}(t)dt \\ \int_0^\infty \overline{H_{1'}}(t)dt \\ \vdots \\ \int_0^\infty \overline{H_{N-1'}}(t)dt \\ \int_0^\infty \overline{H_{N'}}(t)dt \end{pmatrix}$$

We also present the stationary distribution $\boldsymbol{\pi}$ of the SMP (Z_t) as follows

$$\pi_i := \frac{\rho_i m_i}{m}$$

where the mean sojourn time of the system is

$$m := \sum_{i \in E} \rho_i m_i$$

Here we will suppose that the semi-Markov kernel (Q_{ij}) has derivatives (Radon-Nikodym)

$$q_{ij}(t) := \frac{\mathrm{d}}{dt} Q_{ij}(t)$$

The Markov renewal function is expressed as

$$\psi(t) := (I - Q)^{(-1)}(t)$$

If we put the Markov renewal function ψ in bloc matrix form, following the partition U and D of E, we have

$$\psi(t) = \begin{pmatrix} \psi_0 & \psi_{01} \\ \psi_{10} & \psi_1 \end{pmatrix} (t)$$

The reliability function of the system is:

$$R(t) = \boldsymbol{\alpha_0}(I - Q_0)^{(-1)} * \overline{H_0}(t)$$

The instantaneous availability is

$$A(t) = \boldsymbol{\alpha}(I - Q)^{(-1)} * \overline{H}(t)$$

Mean times:

$$MTTF = \boldsymbol{\alpha_0}(I - P_0)^{-1}\boldsymbol{m_0}$$
$$MTTR = \boldsymbol{\alpha_1}(I - P_1)^{-1}\boldsymbol{m_1}$$
$$MUT = \frac{\boldsymbol{\pi_0 m_0}}{\boldsymbol{\pi_1} P_{10} \mathbb{1}_r}$$
$$MDT = \frac{\boldsymbol{\pi_1 m_1}}{\boldsymbol{\pi_0} P_{01} \mathbb{1}_{d-r}}$$

Rate of Occurrence of failure:

$$ro(t) = \alpha_0 \psi_0 * q_{01}(t) \mathbb{1}_{d-r} + \alpha_1 \psi_{10} * q_{01}(t) \mathbb{1}_{d-r}$$

After algebra convolution calculation, we have some final results presented here.

Reliability $R(t) = \sum_{i=0}^{N} \alpha_{i'} \overline{H_{0'}(t)} + \sum_{i=1}^{N} \alpha_{i'} Q_{i'0'} * \overline{H_{i'}(t)}$

Availability

$$A(t) = \alpha_{0'}(det(I-Q))^{(-1)} * (\overline{H_{0'}} + \sum_{j=1}^{N} X_{0j} * \overline{H_{j'}}) + \sum_{i=1}^{N} \alpha_{i'}(det(I-Q))^{(-1)} * (\sum_{j=0}^{i-1} X_{ij} * \overline{H_{j'}} + \sum_{i=1}^{N}(1 - X_{ij}) * \overline{H_{j'}}) + \alpha_{0''}(det(I-Q))^{(-1)} * (\frac{Q_{0'0''}}{1-P_{0'0'}} * \overline{H_{0''}} + \sum_{j=1}^{N-1} Q_{j'j''} * X_{0j} * \overline{H_{j''}}) + \sum_{i=1}^{N-1} \alpha_{i''}(det(I-Q))^{(-1)} * \frac{Q_{0'0''}}{1-P_{0'0'}} * X_{i0} * \overline{H_{0''}} + \sum_{i=1}^{N-1} \alpha_{i''}(det(I-Q))^{(-1)} * Q_{j'j''} * (\sum_{j=0}^{i-1} X_{ij} * \overline{H_{j''}} + \sum_{i=1}^{N}(1 - X_{ij}) * \overline{H_{j''}})$$

Stationary Availability $\overline{A} = \sum_{i \in U} \pi_i = \frac{\sum_{i \in U} \rho_i m_i}{m}$

Mean Time to Failure $MTTF = \alpha_{0'} m_{0'} + \sum_{i=1}^{N} \alpha_{i'0'}(m_{0'} P_{i'0'} + m_{i'})$

Mean Time to Repair $MTTR = \sum_{i=0}^{N-1} \alpha_{i''} m_{i''}$

Mean Up Time $MUT = \frac{\sum_{i=0}^{N} \rho_{i'} m_{i'}^2}{\sum\limits_{i=0}^{N-1} \rho_{i''} m_{i''}}$

Mean Down Time $MDT = \frac{\sum\limits_{i=0}^{N-1} \rho_{i''} m_{i''}^2}{\sum\limits_{i=0}^{N} \rho_{i'} m_{i'}}$

If the system begins with perfect state, we have

Reliability R(t) $= \overline{H_{0'}} = 1 - \frac{Q_{0'0''}(t)}{1-P_{0'0'}}$

Availability $A(t) = (det(I - Q))^{(-1)} * (\overline{H_{0'}} + \sum\limits_{j=1}^{N} X_{0j} * \overline{H_{j'}})$

Mean Time to Failure $MTTF = m_{0'}$

Rate of Occurrence of failure

$$ro(t) = (det(I - Q))^{(-1)} * \left(\frac{q_{0'0''}}{1 - P_{0'0'}} + \sum_{j=1}^{N-1} X_{0j} * q_{j'j''} \right)$$

$$\lim_{t\to\infty} ro(t) = \frac{\sum\limits_{i\in U} \rho_i P_{i'i''}}{m}$$

5. Numerical Examples

Example 1

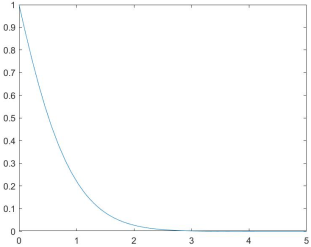
The lifetime of system follows Weilbull distribution $F \sim W(\alpha, \beta)$, same distribution for repair duration $G \sim W(\alpha_1, \beta_1)$

Parameters	Value
λ	0.02/h
p	0.87
α	1
β	1.5
α_1	1
β_1	0.9
a	1.25
b	0.9
N	3

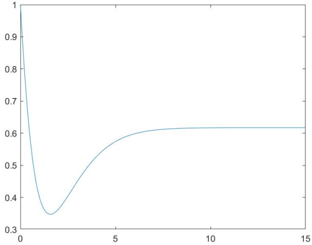
$$\pi = \left(\begin{array}{cccc|ccc} 0.3273 & 0.2870 & 0.0031 & 0.0000 & 0.3770 & 0.0055 & 0.0001 \end{array} \right)$$

$$\boldsymbol{m} = \begin{pmatrix} 0.8682 \\ 0.7612 \\ 0.6107 \\ 0.4896 \\ \hline 1.0000 \\ 1.1111 \\ 1.2346 \end{pmatrix},$$

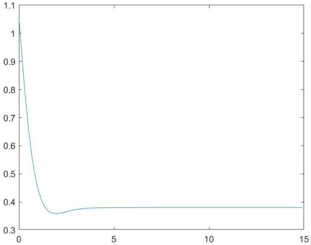
m	0.8682
$MTTF$	0.8682
$MTTR$	0
MUT	1.3185
MDT	1.1574
$\overline{A}$	0.6174
$\lim_{t\to\infty} ro(t)$	0.3820



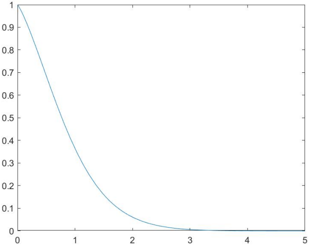
(a) Reliability with perfect initial state



(b) Availability with perfect initial state



(c) ROCOF with perfect initial state



(d) Reliability with Uniform initial law

Figure: Example 1

The lifetime of system follows PH distribution with representation $U_i \sim Ph(\alpha, a^i \boldsymbol{T})$, and same distribution for repair time $D_i \sim Ph(\beta, b^i \boldsymbol{S})$

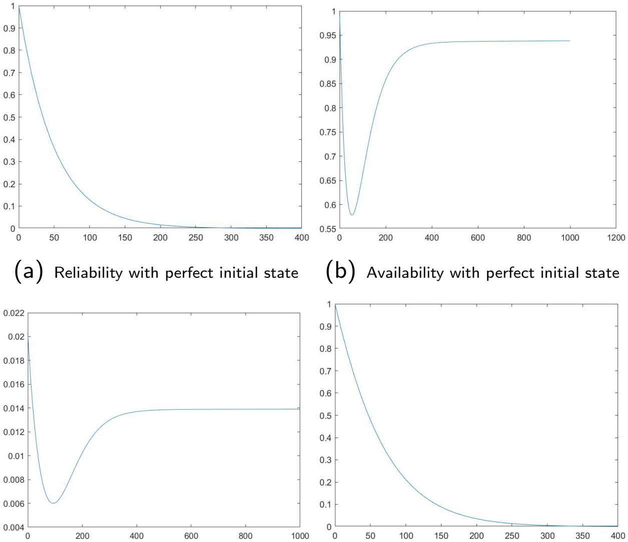
Parameters	Value
λ	0.02/h
p	0.87
α	[1 0 0]
β	[1 0 0]
a	1.25
b	0.9
N	3

$$\boldsymbol{T} = \begin{pmatrix} -0.001 & 0.001 & 0 \\ 0 & -0.08 & 0.08 \\ 0 & 0 & -0.01 \end{pmatrix}, \quad \boldsymbol{S} = \begin{pmatrix} -0.5 & 0.5 & 0 \\ 0.01 & -0.08 & 0.07 \\ 0.005 & 0 & -0.2 \end{pmatrix}$$

$$\boldsymbol{\pi} = \left(\begin{array}{cccc|ccc} 0.2630 & 0.2642 & 0.2235 & 0.1869 & 0.0220 & 0.0208 & 0.0196 \end{array} \right)$$

$$\boldsymbol{m} = \begin{pmatrix} 48.8302 \\ 49.0574 \\ 48.6301 \\ 48.0516 \\ \hline 4.0817 \\ 4.5352 \\ 5.0391 \end{pmatrix},$$

m	30.1861
$MTTF$	48.8302
$MTTR$	0
MUT	731.2785
MDT	0.4177
$\overline{A}$	0.93866
$\lim_{t \to \infty} ro(t)$	0.0139



(a) Reliability with perfect initial state

(b) Availability with perfect initial state

(c) ROCOF with perfect initial state

(d) Reliability with Uniform initial law

Figure: Example 2

6. Conclusions

Conclusion

- Extension of state space allows us to obtain reliability indicators in a standard way.
- Convolution algebra allows us to obtain closed form solution for reliability and non stationary indicators.
- Consideration of duration time of replacement in the future.
- Consideration of repairman replacement in the future.

Bibliography

[Pérez-Ocón et Torres-Castro (2002)] R. Pérez-Ocón and I. Torres-Castro, “A reliability semi-markov model involving geometric processes,” *Applied Stochastic Models in Business and Industry*, vol. 18, no. 2, pp. 157–170, 2002.

[Ravichandran (2001)] N. Ravichandran, *Stochastic methods in reliability theory*. *John Wiley & Sons*, 1990.

[Limnios (2001)] N. Limnios and G. Oprisan, Semi-Markov processes and reliability. *Springer Science & Business Media.*, 2001

[Lam (2007)] Y. Lam, The geometric process and its applications. *World Scientific*, 2007.

[Yeh (1997)] L. Yeh, "The rate of occurrence of failures," *Journal of Applied Probability*, vol. 34, no. 1, pp. 234–247, 1997.

[Limnios (2012)] N. Limnios, "Reliability measures of semi-markov systems with general state space," *Methodology and Computing in Applied Probability*, vol. 14, no. 4, pp. 895–917, 2012.

[Limnios (2010)] N. Limnios, "Semi-markov processes and hidden models," *Wiley Encyclopedia of Operations Research and Management Science*, 2010.

[Ouhbi et Limnios (2002)] B. Ouhbi and N. Limnios, "The rate of occurrence of failures for semi-markov processes and estimation," *Statistics & probability letters*, vol. 59, no. 3, pp. 245–255, 2002.